FACTORIALITY OF q-DEFORMED ARAKI-WOODS VON NEUMANN ALGEBRAS

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Abstract. It is proved that the q-Araki-Woods von Neumann algebras $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ for $q \in (-1, 1)$ are factors if $dim(\mathcal{H}_\mathbb{R}) \geq 3$.

1. introduction

Combining Shlyakhtenko's functor in [Sh97] and the q-Gaussian functor of Bożejko and Speicher in [BS91], Hiai in [Hi03] studied a class of von Neumann algebras which model deformed (canonical) commutation relations (see [FB70]) and yield an interpolation between the Bosonic and Fermionic statistics. These von Neumann algebras are called q-deformed Araki-Woods von Neumann algebras. Recently, there has been a growing interest in q-deformed Araki-Woods von Neumann algebras. These von Neumann algebras depend on three components, namely (i) a real Hilbert space $\mathcal{H}_\mathbb{R}$, (ii) a strongly continuous orthogonal representation (U_t) of $\mathbb{R}$ on $\mathcal{H}_\mathbb{R}$, (iii) a parameter $q \in (-1, 1)$, and are denoted by $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$.

Deciding the factoriality of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ is a non trivial problem and is open for a long time. It was addressed by Hiai in [Hi03] and answered in the affirmative when the almost periodic part of (U_t) is infinite dimensional. Nelson extended the techniques of free monotone transport in the non tracial set up and established the factoriality of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ when $dim(\mathcal{H}_\mathbb{R}) \geq 2$, (U_t) is arbitrary and when q is close to 0 [Ne15] by showing that $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)'' \cong \Gamma_0(\mathcal{H}_\mathbb{R}, U_t)''$, the latter is a factor (free Araki-Woods factor). Recently, we studied generator masas in $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$, proved the factoriality of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ for all $q \in (-1, 1)$ and determined the S-invariant of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ under the hypothesis that (U_t) is non ergodic or weakly mixing and $dim(\mathcal{H}_\mathbb{R}) \geq 2$ [BM16]. Shortly after, Skalski and Wang provided a different proof of factoriality of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ under the same hypothesis as ours [SW16].

In [BM16] we exploited the existence of a fixed vector of (U_t) to produce a masa of the ambient algebra, the former living inside the centralizer of the q-quasi free state. The aforesaid fact was not only crucial to decide the factoriality of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ (when $dim(\mathcal{H}_\mathbb{R}) \geq 2$), but also forced that the centralizer of the q-quasi free state is an irreducible subfactor of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$, thereby facilitating the computation of the S-invariant. Thus, the factoriality of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ is open only in the case when $dim(\mathcal{H}_\mathbb{R})$ is *finite and even and* (U_t) *is ergodic* (see §3 [Hi03], §6 [BM16]).

In this paper, we prove that $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ is a factor when $dim(\mathcal{H}_\mathbb{R}) \geq 4$ and even (regarding ∞ as an even number) for all $q \in (-1, 1)$. Our proof does not assume that $dim(\mathcal{H}_\mathbb{R})$ is finite. Thus, combining the results in [BM16, Hi03] and Thm. 3.3 (of this paper), the factoriality of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ remains open only in the case when $dim(\mathcal{H}_\mathbb{R}) = 2$ and (U_t) is ergodic.

It is to be noted that our results in [BM16] heavily depended on the existence of conditional expectations onto generator masas. In the current scenario, we lack conditional expectations onto generating abelian algebras (which are probably masas) and thus fail

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to meaningfully analyze the GNS space as a bimodule over these abelian algebras as in [BM16]. Instead, we analyze the GNS space as appropriate A-B bimodules, where A, B are subalgebras of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ and where B has appropriate conditional expectation and we show that there is 'enough mixing' to affirmatively conclude the factoriality of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ when $dim(\mathcal{H}_\mathbb{R}) \geq 4$ and (U_t) is ergodic but not weak mixing. Thus, the proofs pertaining to the factoriality in [BM16] and this paper are unified in spirit. This proof was known to us for quite some time but we deferred writing it as we were unable to determine the S-invariant in this case. One of the major obstructions to compute the S-invariant in this case is the lack of knowledge of polar decomposition of a q-circular operator. But several colleagues has encouraged us to write this proof.

The layout of this paper is as follows. In §2, we collect all the necessary facts and formulas involving various properties of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ which are required for the analysis. In §3, we study bimodules and establish the factoriality of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$. We needed some standard facts on bimodules, for some of which we lagged references and hence jotted them down in the form of an appendix.

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2. q-Araki-Woods von Neumann algebras: Basic Facts

In this section, we collect some facts about the q-deformed (free) Araki-Woods von Neumann algebras constructed by Hiai in [Hi03] that will be indispensable for our purpose. For detailed exposition, we refer the interested readers to [Hi03, Sh97]. Some of the facts we need were proved in [BM16] and we will recall those without proofs. This section has substantial overlap with §2 and §3 of [BM16] to keep this paper self-contained. As a convention (following [Sh97, Hi03]), in §2 and §3 we assume that inner products are linear in the second variable.

Let $\mathcal{H}_\mathbb{R}$ be a real Hilbert space and let $t \mapsto U_t$, $t \in \mathbb{R}$, be a strongly continuous orthogonal representation of $\mathbb{R}$ on $\mathcal{H}_\mathbb{R}$. Let $\mathcal{H}_\mathbb{C} = \mathcal{H}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ denote the complexification of $\mathcal{H}_\mathbb{R}$. Denote the inner product and the norm on $\mathcal{H}_\mathbb{C}$ by $\langle \cdot, \cdot \rangle_{\mathcal{H}_\mathbb{C}}$ and $\|\cdot\|_{\mathcal{H}_\mathbb{C}}$ respectively. Identify $\mathcal{H}_\mathbb{R}$ in $\mathcal{H}_\mathbb{C}$ by $\mathcal{H}_\mathbb{R} \otimes 1$. Thus, $\mathcal{H}_\mathbb{C} = \mathcal{H}_\mathbb{R} + i\mathcal{H}_\mathbb{R}$, and as a real Hilbert space the inner product of $\mathcal{H}_\mathbb{R}$ in $\mathcal{H}_\mathbb{C}$ is given by $\Re\langle \cdot, \cdot \rangle_{\mathcal{H}_\mathbb{C}}$. Consider the bounded anti-linear operator $\mathcal{J} : \mathcal{H}_\mathbb{C} \to \mathcal{H}_\mathbb{C}$ given by $\mathcal{J}(\xi + i\eta) = \xi - i\eta$, $\xi, \eta \in \mathcal{H}_\mathbb{R}$, and note that $\mathcal{J}\xi = \xi$ for $\xi \in \mathcal{H}_\mathbb{R}$. Moreover,

$$\langle \xi, \eta \rangle_{\mathcal{H}_\mathbb{C}} = \overline{\langle \eta, \xi \rangle}_{\mathcal{H}_\mathbb{C}} = \langle \eta, \mathcal{J}\xi \rangle_{\mathcal{H}_\mathbb{C}}, \text{ for all } \xi \in \mathcal{H}_\mathbb{C}, \eta \in \mathcal{H}_\mathbb{R}.$$

Linearly extend $t \mapsto U_t$ from $\mathcal{H}_\mathbb{R}$ to a strongly continuous one parameter group of unitaries in $\mathcal{H}_\mathbb{C}$ and denote the extensions by U_t for each t with abuse of notation. Let A denote the analytic generator. Then A is positive, nonsingular and self-adjoint. It is easy to see that $\mathcal{J}A = A^{-1}\mathcal{J}$. Introduce a new inner product on $\mathcal{H}_\mathbb{C}$ by $\langle \xi, \eta \rangle_U = \langle \frac{2}{1+A^{-1}}\xi, \eta \rangle_{\mathcal{H}_\mathbb{C}}$, $\xi, \eta \in \mathcal{H}_\mathbb{C}$, and let $\|\cdot\|_U$ denote the associated norm on $\mathcal{H}_\mathbb{C}$. Let $\mathcal{H}$ denote the complex Hilbert space obtained by completing $(\mathcal{H}_\mathbb{C}, \|\cdot\|_U)$. The inner product and norm of $\mathcal{H}$ will respectively be denoted by $\langle \cdot, \cdot \rangle_U$ and $\|\cdot\|_U$ as well. Then, $(\mathcal{H}_\mathbb{R}, \|\cdot\|_{\mathcal{H}_\mathbb{C}}) \ni \xi \overset{i}{\mapsto} \xi \in (\mathcal{H}_\mathbb{C}, \|\cdot\|_U) \subseteq (\mathcal{H}, \|\cdot\|_U)$, is an isometric embedding of the real Hilbert space $\mathcal{H}_\mathbb{R}$ in $\mathcal{H}$ (in the sense of [Sh97]). With abuse of notation, identify $\mathcal{H}_\mathbb{R}$ with its image $i(\mathcal{H}_\mathbb{R})$. Then, $\mathcal{H}_\mathbb{R} \cap i\mathcal{H}_\mathbb{R} = \{0\}$ and $\mathcal{H}_\mathbb{R} + i\mathcal{H}_\mathbb{R}$ is dense in $\mathcal{H}$ (see pp. 332 [Sh97]).

As A is affiliated to $vN(U_t : t \in \mathbb{R})$, so note that

$$\langle U_t\xi, U_t\eta \rangle_U = \langle \xi, \eta \rangle_U, \text{ for } \xi, \eta \in \mathcal{H}_\mathbb{C}. \tag{1}$$

Consequently, (U_t) extends to a strongly continuous unitary representation $(\widetilde{U}_t)$ of $\mathbb{R}$ on $\mathcal{H}$. Let $\widetilde{A}$ be the analytic generator associated to $(\widetilde{U}_t)$, which is obviously an extension of A. From the definition of $\langle \cdot, \cdot \rangle_U$ on $\mathcal{H}_\mathbb{C}$, it follows that if μ is the spectral measure of

A, then $\nu = f\mu$ is the spectral measure of $\widetilde{A}$, where $f(x) = \frac{2x}{1+x}$ for $x \in \mathbb{R}_{\geq 0}$, and by the spectral theorem, the multiplicity functions in the associated direct integrals remain the same. Thus, we have the following:

Proposition 2.1. *Any eigenvector of $\widetilde{A}$ is an eigenvector of A corresponding to the same eigenvalue.*

Since the spectral information of A and $\widetilde{A}$ (and hence of (U_t) and $(\widetilde{U}_t)$) are essentially the same, and $\widetilde{U}_t, \widetilde{A}$ are respectively extensions of U_t, A for all $t \in \mathbb{R}$, so we would now write $\widetilde{A} = A$ and $\widetilde{U}_t = U_t$ for all $t \in \mathbb{R}$. This abuse of notation will cause no confusion.

Following [BS91], the q-Fock space $\mathcal{F}_q(\mathcal{H})$ of $\mathcal{H}$ is constructed as follows for $-1 < q < 1$. Let Ω be a distinguished unit vector in $\mathbb{C}$ usually referred to as the vacuum vector. Denote $\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$, and, for $n \geq 1$, let $\mathcal{H}^{\otimes n} = \operatorname{span}_{\mathbb{C}}\{\xi_1 \otimes \cdots \otimes \xi_n : \xi_i \in \mathcal{H} \text{ for } 1 \leq i \leq n\}$ denote the algebraic tensor products. Let $\mathcal{F}_{fin}(\mathcal{H}) = \operatorname{span}_{\mathbb{C}}\{\mathcal{H}^{\otimes n} : n \geq 0\}$. For $n, m \geq 0$ and $f = \xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}^{\otimes n}$, $g = \zeta_1 \otimes \cdots \otimes \zeta_m \in \mathcal{H}^{\otimes m}$, the association

$$\langle f, g\rangle_q = \delta_{m,n} \sum_{\pi \in S_n} q^{i(\pi)} \langle \xi_1, \zeta_{\pi(1)}\rangle_U \cdots \langle \xi_n, \zeta_{\pi(n)}\rangle_U, \tag{2}$$

where $i(\pi)$ denotes the number of inversions of the permutation $\pi \in S_n$, defines a positive definite sesquilinear form on $\mathcal{F}_{fin}(\mathcal{H})$ and the q-Fock space $\mathcal{F}_q(\mathcal{H})$ is the completion of $\mathcal{F}_{fin}(\mathcal{H})$ with respect to the norm $\|\cdot\|_q$ induced by $\langle \cdot, \cdot\rangle_q$. For $n \in \mathbb{N}$, let $\mathcal{H}^{\otimes_q n} = \overline{\mathcal{H}^{\otimes n}}^{\|\cdot\|_q}$. For our purposes, it is important to note that $\langle \cdot, \cdot\rangle_q$ and $\langle \cdot, \cdot\rangle_0$ are equivalent on $\mathcal{H}^{\otimes n}$ and $\langle \cdot, \cdot\rangle_0$ is the inner product of the standard tensor product.

The following norm inequalities will be useful (c.f. [BKS97], [BS91], and [Ri05]):

• If $\xi \in \mathcal{H}$ and $\|\xi\|_U = 1$, then

$$\left\|\xi^{\otimes n}\right\|_q^2 = [n]_q!, \tag{3}$$

where $[n]_q := 1 + q + \cdots + q^{(n-1)}$, $[n]_q! := \prod_{j=1}^n [j]_q$, for $n \geq 1$, and $[0]_q := 0$, $[0]_q! := 1$ by convention.

• If $\xi_1, \cdots, \xi_n, \xi \in \mathcal{H}$ with $\|\xi_j\|_U = \|\xi\|_U = 1$ for all $1 \leq j \leq n$, then the following estimate holds:

$$\left\|\xi_1 \otimes \cdots \otimes \xi_n \otimes \xi^{\otimes m}\right\|_q \leq C_q^{\frac{n}{2}} \sqrt{[m]_q!}, \; m \geq 0, \tag{4}$$

where $C_q = \prod_{i=1}^\infty \frac{1}{(1-|q|^i)}$.

For $\xi \in \mathcal{H}$, the left q-creation and q-annihilation operators on $\mathcal{F}_q(\mathcal{H})$ are respectively defined by:

$$\begin{aligned}
&c_q(\xi)\Omega = \xi, \\
&c_q(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n, \text{ and,} \\
&c_q(\xi)^*\Omega = 0, \\
&c_q(\xi)^*(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{i=1}^n q^{i-1} \langle \xi, \xi_i\rangle_U \xi_1 \otimes \cdots \otimes \xi_{i-1} \otimes \xi_{i+1} \otimes \cdots \otimes \xi_n,
\end{aligned} \tag{5}$$

where $\xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}^{\otimes_q n}$ for $n \geq 1$. The operators $c_q(\xi), c_q(\xi)^* \in \mathbf{B}(\mathcal{F}_q(\mathcal{H}))$ and they are adjoints of each other. We have:

Lemma 2.2. *Let $\xi, \xi_i, \eta_j \in \mathcal{H}$, for $1 \leq i \leq n$, $1 \leq j \leq m$. Then,*

$$\begin{aligned}
&c_q(\xi)^*\big((\xi_1 \otimes \cdots \otimes \xi_n) \otimes (\eta_1 \otimes \cdots \otimes \eta_m)\big) \\
=&\big(c_q(\xi)^*(\xi_1 \otimes \cdots \otimes \xi_n)\big) \otimes (\eta_1 \otimes \cdots \otimes \eta_m) + q^n (\xi_1 \otimes \cdots \otimes \xi_n) \otimes \big(c_q(\xi)^*(\eta_1 \otimes \cdots \otimes \eta_m)\big).
\end{aligned}$$

Consider the C^*-algebra $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t) =: C^*\{s_q(\xi) : \xi \in \mathcal{H}_\mathbb{R}\}$ and the von Neumann algebra $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$, where $s_q(\xi) = c_q(\xi) + c_q(\xi)^*$, $\xi \in \mathcal{H}_\mathbb{R}$. This von Neumann algebra is known as the *q-deformed Araki-Woods von Neumann algebra* (see [Hi03, §3]). The vacuum state $\varphi_{q,U} := \langle \Omega, \cdot\, \Omega\rangle_q$ (also called the *q-quasi free state*), is a faithful normal state of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ and $\mathcal{F}_q(\mathcal{H})$ is the GNS Hilbert space of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ associated to $\varphi_{q,U}$. Thus, $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ acting on $\mathcal{F}_q(\mathcal{H})$ is in standard form. We use the symbols $\langle\cdot,\cdot\rangle_q$ and $\|\cdot\|_q$ respectively to denote the inner product and two-norm of elements of the GNS Hilbert space.

The modular theory of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ associated to $\varphi_{q,U}$ is as follows. Let $J_{\varphi_{q,U}}$ and $\Delta_{\varphi_{q,U}}$ respectively denote the modular conjugation and modular operator associated to $\varphi_{q,U}$ and let $S_{\varphi_{q,U}} = J_{\varphi_{q,U}}\Delta^{\frac{1}{2}}_{\varphi_{q,U}}$. Then, for $n \in \mathbb{N}$,

$$\begin{aligned}
(6)\qquad & J_{\varphi_{q,U}}(\xi_1 \otimes \cdots \otimes \xi_n) = A^{-1/2}\xi_n \otimes \cdots \otimes A^{-1/2}\xi_1,\ \forall\ \xi_i \in \mathcal{H}_\mathbb{R} \cap \mathfrak{D}(A^{-\frac{1}{2}});\\
& \Delta_{\varphi_{q,U}}(\xi_1 \otimes \cdots \otimes \xi_n) = A^{-1}\xi_1 \otimes \cdots \otimes A^{-1}\xi_n,\ \forall\ \xi_i \in \mathcal{H}_\mathbb{R} \cap \mathfrak{D}(A^{-1});\\
& S_{\varphi_{q,U}}(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_n \otimes \cdots \otimes \xi_1,\ \forall\ \xi_i \in \mathcal{H}_\mathbb{R}.
\end{aligned}$$

The modular automorphism group $(\sigma_t^{\varphi_{q,U}})$ of $\varphi_{q,U}$ is given by $\sigma_{-t}^{\varphi_{q,U}} = \mathrm{Ad}(\mathcal{F}(U_t))$, where $\mathcal{F}(U_t) = id \oplus \oplus_{n\geq 1} U_t^{\otimes_q n}$, for all $t \in \mathbb{R}$. In particular,

$$(7)\qquad \sigma_{-t}^{\varphi_{q,U}}(s_q(\xi)) = s_q(U_t\xi),\ \text{for all } \xi \in \mathcal{H}_\mathbb{R}.$$

Consider the set $\mathcal{H}'_\mathbb{R} = \{\xi \in \mathcal{H} : \langle \xi, \eta\rangle_U \in \mathbb{R}\ \forall\ \eta \in \mathcal{H}_\mathbb{R}\}$. Note that $\overline{\mathcal{H}'_\mathbb{R} + i\mathcal{H}'_\mathbb{R}} = \mathcal{H}$ and $\mathcal{H}'_\mathbb{R} \cap i\mathcal{H}'_\mathbb{R} = \{0\}$. Let $\zeta \in \mathfrak{D}(A^{-1/2}) \cap \mathcal{H}_\mathbb{R}$. Note that for all $\eta \in \mathcal{H}_\mathbb{R}$, one has

$$\begin{aligned}
(8)\qquad \langle A^{-1/2}\zeta, \eta\rangle_U &= \langle \frac{2A^{-1/2}}{1+A^{-1}}\zeta, \eta\rangle_{\mathcal{H}_\mathbb{C}} = \langle \eta, \mathcal{J}\frac{2A^{-1/2}}{1+A^{-1}}\zeta\rangle_{\mathcal{H}_\mathbb{C}} = \langle \eta, \frac{2A^{1/2}}{1+A}\zeta\rangle_{\mathcal{H}_\mathbb{C}}\\
&= \langle \frac{2}{1+A^{-1}}\eta, A^{-1/2}\zeta\rangle_{\mathcal{H}_\mathbb{C}} = \langle \eta, A^{-1/2}\zeta\rangle_U.
\end{aligned}$$

From Eq. (8), it follows that $A^{-1/2}\zeta \in \mathcal{H}'_\mathbb{R}$ for all $\zeta \in \mathfrak{D}(A^{-\frac{1}{2}}) \cap \mathcal{H}_\mathbb{R}$. Also note that for $\eta, \xi \in \mathfrak{D}(A^{-1}) \cap \mathcal{H}_\mathbb{R}$, one has

$$\begin{aligned}
(9)\qquad \langle \eta, \xi\rangle_U &= \langle \frac{2}{1+A^{-1}}\eta, \xi\rangle_{\mathcal{H}_\mathbb{C}} = \langle \xi, \mathcal{J}\frac{2}{1+A^{-1}}\eta\rangle_{\mathcal{H}_\mathbb{C}} = \langle \xi, \frac{2}{1+A}\eta\rangle_{\mathcal{H}_\mathbb{C}}\\
&= \langle \frac{2}{1+A^{-1}}\xi, A^{-1}\eta\rangle_{\mathcal{H}_\mathbb{C}} = \langle \xi, A^{-1}\eta\rangle_U = \langle A^{-\frac{1}{2}}\xi, A^{-\frac{1}{2}}\eta\rangle_U.
\end{aligned}$$

Now for $\xi \in \mathcal{H}$, define the right creation operator $r_q(\xi)$ on $\mathcal{F}_q(\mathcal{H})$ by

$$\begin{aligned}
(10)\qquad & r_q(\xi)\Omega = \xi,\\
& r_q(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_1 \otimes \cdots \otimes \xi_n \otimes \xi,\ \ \xi_i \in \mathcal{H}, n \geq 1.
\end{aligned}$$

Clearly, $r_q(\xi) = \jmath c_q(\xi)\jmath^*$, where $\jmath : \mathcal{F}_q(\mathcal{H}) \to \mathcal{F}_q(\mathcal{H})$ is the unitary defined by

$$\begin{aligned}
(11)\qquad & \jmath(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_n \otimes \cdots \otimes \xi_1,\ \text{ where } \xi_i \in \mathcal{H} \text{ for all } 1 \leq i \leq n, n \geq 1,\\
& \jmath(\Omega) = \Omega.
\end{aligned}$$

Therefore, $r_q(\xi) \in \mathbf{B}(\mathcal{F}_q(\mathcal{H}))$ and its adjoint $r_q(\xi)^*$ is given by

(12)

$$\begin{aligned}
& r_q(\xi)^*\Omega = 0,\\
& r_q(\xi)^*(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{i=1}^n q^{n-i}\langle \xi, \xi_i\rangle_U \xi_1 \otimes \cdots \otimes \xi_{i-1} \otimes \xi_{i+1} \otimes \cdots \otimes \xi_n,\ \ \xi_i \in \mathcal{H}, n \geq 1.
\end{aligned}$$

Write $d_q(\xi) = r_q(\xi) + r_q(\xi)^*$, $\xi \in \mathcal{H}$. Then:

Theorem 2.3 (cf. Thm. 3.3, [Sh97])**.** *Suppose $\xi \in \mathfrak{D}(A^{-1}) \cap \mathcal{H}_{\mathbb{R}}$. Then $J_{\varphi_{q,U}} s_q(\xi) J_{\varphi_{q,U}} = d_q(A^{-\frac{1}{2}}\xi)$. Moreover, $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)' = \{d_q(\xi) : \xi \in \mathcal{H}'_{\mathbb{R}}\}''$.*

In this paper, we are interested in the factoriality of $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ and the orthogonal representation remain arbitrary but fixed. Thus, to reduce notation, we will write $M_q = \Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ and $\varphi = \varphi_{q,U}$. We will also denote $J_{\varphi_{q,U}}$ by J and $\Delta_{\varphi_{q,U}}$ by Δ. Write $\mathcal{Z}(M_q) = M_q \cap M_q'$. We say that a vector $\xi \in \mathcal{H}_{\mathbb{R}}$ is *analytic*, if $s_q(\xi)$ is analytic for (σ_t^{φ}). As Ω is separating for both M_q and M_q', so for $\zeta \in M_q\Omega$ and $\eta \in M_q'\Omega$ there exist unique $x_\zeta \in M_q$ and $x'_\eta \in M_q'$ such that $\zeta = x_\zeta\Omega$ and $\eta = x'_\eta\Omega$. In this case, we will write

$$s_q(\zeta) = x_\zeta \text{ and } d_q(\eta) = x'_\eta. \tag{13}$$

Thus, for example, as $\xi \in M_q\Omega$ for every $\xi \in \mathcal{H}_{\mathbb{R}}$, so $s_q(\xi + i\eta) = s_q(\xi) + is_q(\eta)$ for all $\xi, \eta \in \mathcal{H}_{\mathbb{R}}$.
Caution: Note that $c_q(\xi)$ and $r_q(\xi)$ are bounded operators for all $\xi \in \mathcal{H}$. Write

$$\widetilde{s}_q(\xi) = c_q(\xi) + c_q(\xi)^* \text{ and } \widetilde{d}_q(\xi) = r_q(\xi) + r_q(\xi)^*, \ \xi \in \mathcal{H}.$$

Note that if $\xi \in \mathcal{H}_{\mathbb{R}}$, then $\widetilde{s}_q(\xi) = s_q(\xi)$, and if $\xi \in \mathcal{H}'_{\mathbb{R}}$ then $\widetilde{d}_q(\xi) = d_q(\xi)$. If $\xi = \xi_1 + i\xi_2$ for $\xi_1, \xi_2 \in \mathcal{H}_{\mathbb{R}}$ and $\xi_2 \neq 0$, then note that $\widetilde{s}_q(\xi) \neq s_q(\xi)$.

Lemma 2.4. *The following hold.*

1. *The vector $\xi_1 \otimes \cdots \otimes \xi_n \in M_q\Omega$ for any $\xi_i \in \mathcal{H}_{\mathbb{R}}$, $1 \leq i \leq n$ and $n \in \mathbb{N}$.*
2. *The vector $\xi_1 \otimes \cdots \otimes \xi_n \in M_q'\Omega$ for any $\xi_i \in \mathfrak{D}(A^{-\frac{1}{2}}) \cap \mathcal{H}_{\mathbb{R}}$, $1 \leq i \leq n$ and $n \in \mathbb{N}$.*
3. *$\mathcal{H}_{\mathbb{R}}$ has an orthonormal basis with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}_{\mathbb{C}}}$ comprising of analytic vectors.*

Fix $\xi \in \mathcal{H}_{\mathbb{R}}$ with $\|\xi\|_U = 1$. By Eq. (1.2) of [Hi03], the moments of the operator $s_q(\xi)$ with respect to the q-quasi free state $\varphi(\cdot) = \langle \Omega, \cdot\Omega \rangle_q$ is given by

$$\varphi(s_q(\xi)^n) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \sum_{\mathcal{V} = \{\pi(r), \kappa(r)\}_{1 \leq r \leq \frac{n}{2}}} q^{c(\mathcal{V})}, & \text{if } n \text{ is even,} \end{cases}$$

where the summation is taken over all pair partitions $\mathcal{V} = \{\pi(r), \kappa(r)\}_{1 \leq r \leq \frac{n}{2}}$ of $\{1, 2, \cdots, n\}$ with $\pi(r) < \kappa(r)$ and $c(\mathcal{V})$ is the number of crossings of $\mathcal{V}$, i.e,

$$c(\mathcal{V}) = \#\{(r, s) : \pi(r) < \pi(s) < \kappa(r) < \kappa(s)\}.$$

The distribution of $s_q(\xi)$ does not depend on the group (U_t). This distribution obeys the semicircular law ν_q which is absolutely continuous with respect to the uniform measure supported on the interval $[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}]$; the associated orthogonal polynomials are q-Hermite polynomials H_n^q, $n \geq 0$. Thus, M_ξ is diffuse and $\{H_n^q(s_q(\xi))\Omega : n \geq 0\}$, is a total orthogonal set of vectors in $\overline{M_\xi\Omega}^{\|\cdot\|_q}$. Write $\mathcal{E}_\xi = \{\xi^{\otimes n} : n \geq 0\}$, where $\xi^{\otimes 0} = \Omega$ by convention. Also note that $\overline{M_\xi\Omega}^{\|\cdot\|_q} = \overline{\text{span } \mathcal{E}_\xi}^{\|\cdot\|_q}$ as $\xi^{\otimes n} = H_n^q(s_q(\xi))\Omega$ for all $n \geq 0$.

It is to be noted that for $\xi \in \mathcal{H}_{\mathbb{R}}$ with $\|\xi\|_U = 1$, there does not exist any φ-preserving normal conditional expectation (even operator valued weight) on M_ξ unless $U_t\xi = \xi$ for all $t \in \mathbb{R}$ (see Thm. 4.2 [BM16]). This is the precise problem that poses challenges to settle many issues including factoriality of M_q in all attempts that has been made so far. We suspect that M_ξ is a masa. Had that been so, many more basic properties of M_q could have been handled comfortably. However, there are no techniques available in the literature to check if an abelian algebra is a masa in the absence of appropriate conditional expectation.

3. Factoriality

In this section, we address the issue of factoriality of M_q. Combining the results of [BM16] and [Hi03], the factoriality of M_q is open only in the case $dim(\mathcal{H}_{\mathbb{R}})$ is finite and even.

Since $t \mapsto U_t$, $t \in \mathbb{R}$, is a strongly continuous orthogonal representation of $\mathbb{R}$ on the real Hilbert space $\mathcal{H}_{\mathbb{R}}$, so there is a unique decomposition (c.f. [Sh97]),

$$(\mathcal{H}_{\mathbb{R}}, U_t) = \left(\bigoplus_{j=1}^{N_1} (\mathbb{R}, \mathrm{id}) \right) \oplus \left(\bigoplus_{k=1}^{N_2} (\mathcal{H}_{\mathbb{R}}(k), U_t(k)) \right) \oplus (\widetilde{\mathcal{H}}_{\mathbb{R}}, \widetilde{U}_t), \tag{14}$$

where $0 \leq N_1, N_2 \leq \aleph_0$,

$$\mathcal{H}_{\mathbb{R}}(k) = \mathbb{R}^2, \quad U_t(k) = \begin{pmatrix} \cos(t \log \lambda_k) & -\sin(t \log \lambda_k) \\ \sin(t \log \lambda_k) & \cos(t \log \lambda_k) \end{pmatrix}, \ \lambda_k > 1, \tag{15}$$

and $(\widetilde{\mathcal{H}}_{\mathbb{R}}, \widetilde{U}_t)$ corresponds to the weakly mixing component of the orthogonal representation; thus $\widetilde{\mathcal{H}}_{\mathbb{R}}$ is either 0 or infinite dimensional.

In this section, we assume that the strongly continuous real orthogonal representation $t \mapsto U_t$, $t \in \mathbb{R}$, does not admit any non trivial fixed vector i.e., it is a ergodic representation and also assume that there are no weakly mixing components of the representation, though we would not make any specific use of lack of weak mixing. Thus, we suppose that $N_1 = 0$ and $\widetilde{\mathcal{H}}_{\mathbb{R}} = 0$. This reduction is because the other cases to decide the factoriality has been addressed in [BM16, Hi03]. We will further assume that $N_2 \geq 2$.

Let $\xi_{2k-1} = 0 \oplus \cdots \oplus 0 \oplus \begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus 0 \oplus \cdots \oplus 0 \in \bigoplus_{k=1}^{N_2} \mathcal{H}_{\mathbb{R}}(k)$ and $\xi_{2k} = 0 \oplus \cdots \oplus 0 \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix} \oplus 0 \oplus \cdots \oplus 0 \in \bigoplus_{k=1}^{N_2} \mathcal{H}_{\mathbb{R}}(k)$ be vectors with non zero entries in the k-th position for $1 \leq k \leq N_2$. Denote

$$\zeta_{2k-1} = \frac{\sqrt{\lambda_k + 1}}{2}(\xi_{2k-1} + i\xi_{2k}) \text{ and } \zeta_{2k} = \frac{\sqrt{\lambda_k^{-1} + 1}}{2}(\xi_{2k-1} - i\xi_{2k}),$$

thus $\zeta_{2k-1}, \zeta_{2k} \in \mathcal{H}_{\mathbb{R}}(k) + i\mathcal{H}_{\mathbb{R}}(k)$ form an orthonormal basis of $(\mathcal{H}_{\mathbb{R}}(k) + i\mathcal{H}_{\mathbb{R}}(k), \langle \cdot, \cdot \rangle_U)$ for $1 \leq k \leq N_2$. Fix $1 \leq k \leq N_2$. The analytic generator $A(k)$ of $(U_t(k))$ is given by

$$A(k) = \frac{1}{2} \begin{pmatrix} \lambda_k + \frac{1}{\lambda_k} & i(\lambda_k - \frac{1}{\lambda_k}) \\ -i(\lambda_k - \frac{1}{\lambda_k}) & \lambda_k + \frac{1}{\lambda_k} \end{pmatrix}.$$

Moreover,

$$A(k)\zeta_{2k-1} = \frac{1}{\lambda_k}\zeta_{2k-1} \text{ and } A(k)\zeta_{2k} = \lambda_k \zeta_{2k}.$$

Further, denoting $a_k = \log \lambda_k$, one has

$$\begin{aligned} A^{-\frac{1}{2}}\xi_{2k-1} &= \cosh(\frac{1}{2}a_k)\xi_{2k-1} + i\sinh(\frac{1}{2}a_k)\xi_{2k}, \\ A^{-\frac{1}{2}}\xi_{2k} &= -i\sinh(\frac{1}{2}a_k)\xi_{2k-1} + \cosh(\frac{1}{2}a_k)\xi_{2k}. \end{aligned} \tag{16}$$

Fix k, with $1 \leq k \leq N_2$, and rename the pair $(\xi_{2k-1}, \xi_{2k}) = (\xi_0, \xi_0')$ to distinguish it from other pairs in the enumeration. Let $\mathcal{O} = \cup_{k=1}^{N_2}\{\xi_{2k-1}, \xi_{2k}\}$.

Let $B_0 = vN(s_q(\xi_0), s_q(\xi_0'))$. By Eq. (6), Eq. (7) and Eq. (15), note that B_0 is globally invariant with respect to (σ_t^{φ}), thus there exists a unique φ-preserving faithful normal conditional expectation $\mathbb{E}_{B_0} : M_q \to B_0$ from M_q onto B_0 by a well known theorem of Takesaki [Ta72]. Let e_{B_0} denote the Jones' projection associated to B_0. For $x, y \in$

M_q analytic with respect to (σ_t^{φ}), consider the bounded operator $T_{x,y} : L^2(M_{\xi_0}, \varphi) \to L^2(B_0, \varphi)$ defined by,

$$T_{x,y}(a\Omega) = \mathbb{E}_{B_0}(xay)\Omega, \ a \in M_{\xi_0}. \tag{17}$$

Note that by [Fa00], one has $\|T_{x,y}\| \leq \|x\| \left\| \sigma^{\varphi}_{\frac{i}{2}}(y) \right\|$.

Then, the following kind of mixing holds. The idea of the proof is essentially the same as that of Thm. 5.3 of [BM16] but modifications are necessary.

Lemma 3.1. *Let $x = s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m})$ and $y = s_q(A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_k})$ be such that $\mathbb{E}_{B_0}(x) = 0 = \mathbb{E}_{B_0}(y)$, where $\xi_{i_l}, \xi_{j_{l'}} \in \mathcal{O}$ for $1 \leq l \leq m$ and $1 \leq l' \leq k$. Then, $T_{x,y}$ is a Hilbert-Schmidt operator.*

Proof. First note that x, y are analytic with respect to (σ_t^{φ}). Indeed, by Eq. (15) and Eq. (16) it follows that $s_q(\xi_{2p-1}), s_q(\xi_{2p}), s_q(A^{-\frac{1}{2}}\xi_{2p-1}), s_q(A^{-\frac{1}{2}}\xi_{2p})$ are all analytic with respect to (σ_t^{φ}) for all $1 \leq p \leq N_2$. The proof of Lemma 3.1 of [BM16] and Eq. (16) shows that x, y lies in the $*$-subalgebra generated by $\{s_q(\xi_{2p-1}), s_q(\xi_{2p}) : 1 \leq p \leq N_2\}$, which in turn is contained in the $*$-subalgebra of analytic elements of M_q. Thus, $T_{x,y}$ is bounded.

Further, note that from the discussion in §2 and Eq. (4) it follows that $\frac{1}{\sqrt{[n]_q!}} H_n^q(s_q(\xi_0))\Omega = \frac{1}{\sqrt{[n]_q!}} \xi_0^{\otimes n}$, $n \geq 0$, is an orthonormal basis of $L^2(M_{\xi_0}, \varphi)$. Also note that $A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_k} \in M_q\Omega \cap M_q'\Omega$ from Lemma 2.4 and Eq. (16), and $d_q(A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_k}) \in M_q'$ by Thm. 2.3.

Furthermore, note that since $\xi_{i_1} \otimes \cdots \otimes \xi_{i_m}$, $A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_k} \in L^2(M_q, \varphi) \ominus L^2(B_0, \varphi)$, so at least one of the letters in both $\xi_{i_1} \otimes \cdots \otimes \xi_{i_m}$ and $\xi_{j_1} \otimes \cdots \otimes \xi_{j_k}$ has to be different from both ξ_0 and ξ_0'. This follows from Eq. (2), Eq. (9) and the fact that B_0 is globally invariant with respect to (σ_t^{φ}), which forces that $A^{-\frac{1}{2}}\xi_0, A^{-\frac{1}{2}}\xi_0' \in B_0\Omega \subseteq L^2(B_0, \varphi)$.

From Eq. (17), we have

$$\begin{aligned}
T_{x,y}\left(H_n^q(s_q(\xi_0))\Omega\right) &= \mathbb{E}_{B_0}\left(xH_n^q(s_q(\xi_0))y\right)\Omega \\
&= e_{B_0}\left(xH_n^q(s_q(\xi_0))s_q(A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_k})\Omega\right) \\
&= e_{B_0}\left(xH_n^q(s_q(\xi_0))(A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_k})\right) \text{ (by Eq. (13))} \\
&= e_{B_0}\left(xH_n^q(s_q(\xi_0))d_q(A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_k})\Omega\right) \text{ (by Eq. (13))} \\
&= e_{B_0}\left(xd_q(A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_k})H_n^q(s_q(\xi_0))\Omega\right) \\
&= e_{B_0}\left(xd_q(A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_k})\xi_0^{\otimes n}\right) \\
&= e_{B_0}\left(s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m})d_q(A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_k})\xi_0^{\otimes n}\right), \ n \geq 0.
\end{aligned} \tag{18}$$

Now from Lemma 3.1 of [Hi03], we have

$$s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m}) = \sum\sum q^{\aleph(K,I)} c_q(\xi_{i_{\kappa(1)}}) \cdots c_q(\xi_{i_{\kappa(n_1)}}) c_q(\xi_{i_{\pi(1)}})^* \cdots c_q(\xi_{i_{\pi(n_2)}})^* \text{ and}$$

$$\begin{aligned}
&d_q(A^{-\frac{1}{2}}\xi_{j_1} \otimes \cdots \otimes A^{-\frac{1}{2}}\xi_{j_k}) \\
&\quad = \sum\sum q^{\aleph(K',I')} r_q(A^{-\frac{1}{2}}\xi_{j_{\tilde{\kappa}(1)}}) \cdots r_q(A^{-\frac{1}{2}}\xi_{j_{\tilde{\kappa}(m_1)}}) r_q(A^{-\frac{1}{2}}\xi_{j_{\tilde{\pi}(1)}})^* \cdots r_q(A^{-\frac{1}{2}}\xi_{j_{\tilde{\pi}(m_2)}})^*,
\end{aligned}$$

where the first sum varies over the pairs (n_1, n_2) and (K, I) restricted to the following conditions:

$$\begin{array}{ccc} & & K=\{\kappa(1),\cdots,\kappa(n_1):\kappa(1)\leq\cdots\leq\kappa(n_1)\}, \\ (19) & \begin{array}{c} n_1,n_2\geq 0,\\ n_1+n_2=m; \end{array} \quad \text{and} & I=\{\pi(1),\cdots,\pi(n_2):\pi(1)\leq\cdots\leq\pi(n_2)\}, \\ & & K\cup I=\{1,\cdots,m\}, K\cap I=\emptyset, \end{array}$$

and $\aleph(K,I)=\#\{(r,s):1\leq r\leq n_1, 1\leq s\leq n_2, \kappa(n_1)>\pi(n_2)\}$. Similar is the expression for $d_q(A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_k})$ in terms of $m_1,m_2\geq 0$, $m_1+m_2=k$, $K',I',\aleph(K',I')$, $\tilde{\kappa},\tilde{\pi}$ and is defined analogous to Eq. (19).

We have to show that $\sum_{n=0}^{\infty}\frac{1}{[n]_q!}\left\|T_{x,y}(H_n^q(s_q(\xi_0))\Omega)\right\|_q^2<\infty$. But since $s_q(\xi_{i_1}\otimes\cdots\otimes\xi_{i_m})$ and $d_q(A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_k})$ split as finite sums, so from Eq. (18) it is enough to show that for each fixed $n_1,n_2,m_1,m_2,\kappa,\pi,\tilde{\kappa},\tilde{\pi}$, if

$$\begin{aligned}\zeta_n = e_{B_0}\Big(& c_q(\xi_{i_{\kappa(1)}})\cdots c_q(\xi_{i_{\kappa(n_1)}})c_q(\xi_{i_{\pi(1)}})^*\cdots c_q(\xi_{i_{\pi(n_2)}})^* \\ & \cdot(r_q(A^{-\frac{1}{2}}\xi_{j_{\tilde{\kappa}(1)}})\cdots r_q(A^{-\frac{1}{2}}\xi_{j_{\tilde{\kappa}(m_1)}})r_q(A^{-\frac{1}{2}}\xi_{j_{\tilde{\pi}(1)}})^*\cdots r_q(A^{-\frac{1}{2}}\xi_{j_{\tilde{\pi}(m_2)}})^*)\xi_0^{\otimes n}\Big), n\geq 0,\end{aligned}$$

then $\sum_{n=0}^{\infty}\frac{1}{[n]_q!}\left\|\zeta_n\right\|_q^2<\infty$. Renaming indices, we may write

$$\begin{aligned}\zeta_n = e_{B_0}\Big(& (c_q(\xi_{i_1})\cdots c_q(\xi_{i_l})c_q(\xi_{i_{l+1}})^*\cdots c_q(\xi_{i_m})^*) \\ & \cdot(r_q(A^{-\frac{1}{2}}\xi_{j_1})\cdots r_q(A^{-\frac{1}{2}}\xi_{j_p})r_q(A^{-\frac{1}{2}}\xi_{j_{p+1}})^*\cdots r_q(A^{-\frac{1}{2}}\xi_{j_k})^*\xi_0^{\otimes n}\Big),\ n\geq 0.\end{aligned}$$

Since at least one letter in $\xi_{j_1}\otimes\cdots\otimes\xi_{j_k}$ is different from both ξ_0 and ξ_0', so from Eq. (5), Eq. (10) and Eq. (12) it follows that ζ_n can be non zero only when $\{\xi_{i_{l+1}},\cdots,\xi_{i_m}\}\subseteq\{\xi_0,\xi_0'\}$. Write $\delta=\prod_{w=p+1}^{k}(\delta_{\xi_{j_w},\xi_0}+\delta_{\xi_{j_w},\xi_0'})$. Let $C_0=\prod_{w=p+1}^{k}\langle A^{-\frac{1}{2}}\xi_{j_w},\xi_0\rangle_U$. Hence, from Eq. (10) and Eq. (12), we have

(20)

$$\begin{aligned}\zeta_n &= C_0\delta\prod_{t=n-(k-p)}^{n}(1+q+\cdots+q^t) \\ &\qquad\cdot e_{B_0}\Big(c_q(\xi_{i_1})\cdots c_q(\xi_{i_l})c_q(\xi_{i_{l+1}})^*\cdots c_q(\xi_{i_m})^*\big(\xi_0^{\otimes(n-(k-p))}\otimes A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_p}\big)\Big) \\ &= C_0\delta\frac{[n]_q!}{[n-(k-p)]_q!}e_{B_0}\Big(c_q(\xi_{i_1})\cdots c_q(\xi_{i_l})c_q(\xi_{i_{l+1}})^*\cdots c_q(\xi_{i_m})^*\big(\xi_0^{\otimes(n-(k-p))}\otimes A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_p}\big)\Big).\end{aligned}$$

By the hypothesis at least one letter in $\xi_{j_1}\otimes\cdots\otimes\xi_{j_p}$ is different from both ξ_0 and ξ_0'. Therefore, the constraints for ζ_n to be non zero are at least -

(i) $\xi_{i_r}\in\{\xi_0,\xi_0'\}$, for all $1\leq r\leq l$;

(ii) $\#\{i_r: l+1\leq r\leq m, \xi_{i_r}\notin\{\xi_0,\xi_0'\}\}\geq 1$ (counted with multiplicities),

(iii) $c_q(\xi_{i_1})\cdots c_q(\xi_{i_l})c_q(\xi_{i_{l+1}})^*\cdots c_q(\xi_{i_m})^*(\xi_0^{\otimes(n-(k-p))}\otimes A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_p})$ has to contain an elementary tensor whose letters are in $\{\xi_0,\xi_0'\}$.

From here onwards the arguments are exactly the same as those of Thm. 5.3 [BM16]. But we append it here for the sake of easy reading. By repeated application of Lemma 2.2, one obtains

(21)

$$c_q(\xi_{i_{l+1}})^*\cdots c_q(\xi_{i_m})^*\left(\overbrace{\xi_0^{\otimes(n-(k-p))}}\otimes\overbrace{(A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_p})}\right)$$

$$=c_q(\xi_{i_{l+1}})^*\cdots c_q(\xi_{i_{m-1}})^*\Bigg(\overbrace{\big(c_q(\xi_{i_m})^*\xi_0^{\otimes(n-(k-p))}\big)}\otimes\overbrace{(A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_p})}$$

$$+\,q^{(n-(k-p))}\,\overbrace{\xi_0^{\otimes(n-(k-p))}}\otimes\overbrace{c_q(\xi_{i_m})^*(A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_p})}\Bigg)$$

$$=\sum_{r_1=0}^{1}\cdots\sum_{r_{m-l}=0}^{1}c_{r_1,\cdots,r_{m-l}}\cdot$$

$$\left(\prod_{w=1}^{m-l}\big(c_q(\xi_{i_{l+w}})^*\big)^{(1-r_w)}\right)\xi_0^{\otimes(n-(k-p))}\otimes\left(\prod_{w=1}^{m-l}\big(c_q(\xi_{i_{l+w}})^*\big)^{r_w}\right)(A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_p}),$$

where $c_{r_1,\cdots,r_{m-l}}\in\mathbb{R}$ for $(r_1,\cdots,r_{m-l})\in\{0,1\}^{m-l}$ are calculated as follows.

Given a $(m-l)$-bit string $(r_1,\cdots,r_{m-l})$, let $s_w=\#$ of zeros in $\{r_w,r_{w+1},\cdots,r_{m-l}\}$ for $1\le w\le m-l$. Then, clearly $s_{m-l}=1-r_{m-l}$ and by induction it follows that $s_{m-l-1}=(1-r_{m-l})+(1-r_{m-l-1})$, $\cdots$, $s_1=(1-r_{m-l})+(1-r_{m-l-1})+\cdots+(1-r_1)$. Thus, repeated application of Lemma 2.2 in Eq. (21) entail that

$$\begin{aligned}
c_{r_1,\cdots,r_{m-l}}&=q^{(n-(k-p))\left(\sum_{w=1}^{m-l}r_w\right)-\sum_{w=1}^{m-l}r_ws_w}\\
&=q^{(n-(k-p))\left(\sum_{w=1}^{m-l}r_w\right)-\sum_{w=1}^{m-l}r_w\left((m-l)-w+1-\sum_{w'=w}^{m-l}r_{w'}\right)}\\
&=q^{\left((n-(k-p))-(m-l)-1\right)\left(\sum_{w=1}^{m-l}r_w\right)+\sum_{w=1}^{m-l}wr_w+\sum_{w=1}^{m-l}\left(\sum_{w'=w}^{m-l}r_{w'}\right)r_w}.
\end{aligned}$$

The above formula for $c_{r_1,\cdots,r_{m-l}}$ can be obtained by drawing a binary tree of height $(m-l)$ with weights attached along the edges in such a way that it encodes the tensoring on the left or on the right following Lemma 2.2. It is to be noted that the largest power of q that appears in Eq. (21) is $(n-(k-p))(m-l)$ which appears when $r_w=1$ for all w and the smallest power of q is 0 and it occurs when $r_w=0$ for all w.

Further, notice that since $\#\{i_r : l+1\le r\le m, \xi_{i_r}\notin\{\xi_0,\xi_0'\}\}\ge 1$, i.e., there is at least one r_0 with $l+1\le r_0\le m$ such that $\xi_{i_{r_0}}\perp\xi_0,\xi_0'$ (in $\langle\cdot,\cdot\rangle_U$), so

$$\Big(c_q(\xi_{i_{l+1}})^*\cdots c_q(\xi_{i_{m-1}})^*c_q(\xi_{i_m})^*\Big)\xi_0^{\otimes(n-(k-p))}\otimes\Big(A^{-\frac{1}{2}}\xi_{j_1}\otimes\cdots\otimes A^{-\frac{1}{2}}\xi_{j_p}\Big)=0.$$

Therefore, the expression in Eq. (21) has at most 2^{m-l-1} many non zero terms each with scalar coefficients of the form q^d, where $d\ge((n-(k-p))-(m-l-1))$. Consequently, by Eq. (4), Eq. (5), Eq. (11) and Eq. (20), we conclude that there is a positive constant $K(l,m,p,q)$ independent of n and $N_0\in\mathbb{N}$ such that

$$\|\zeta_n\|_q^2\le K(l,m,p,q)q^{2n}\left(\frac{[n]_{|q|}!}{[n-(k-p)]_{|q|}!}\sqrt{[n-N_0]_q!}\right)^2,\ \text{for all } n>N_0.$$

Define a sequence $\{a_n\}$ of real numbers as follows:

$$a_n=\begin{cases}1, & \text{if } 0\le n\le N_0,\\ \frac{1}{[n]_q!}|q|^{2n}\left(\frac{[n]_{|q|}!}{[n-(k-p)]_{|q|}!}\sqrt{[n-N_0]_q!}\right)^2, & \text{otherwise.}\end{cases}$$

Note that $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=|q|^2<1$. Consequently, by ratio test $\sum_{n\ge1}a_n<\infty$. Since the sequence $\{a_n\}$ eventually dominates the tail of the sequence $\{\frac{1}{[n]_{q!}}\|\zeta_n\|_q^2\}$ modulo a scalar multiple, the proof is complete. $\square$

Theorem 3.2. *Let $x = s_q(\xi_{i_1} \otimes \cdots \otimes \xi_{i_m})$ and $y = s_q(\xi_{j_1} \otimes \cdots \otimes \xi_{j_k})$ be such that $\mathbb{E}_{B_0}(x) = 0 = \mathbb{E}_{B_0}(y)$, where $\xi_{i_l}, \xi_{j_{l'}} \in \mathcal{O}$ for $1 \leq l \leq m$ and $1 \leq l' \leq k$. Then, $T_{x,y}$ is a Hilbert-Schmidt operator.*

Moreover, ${}_{M_{\xi_0}}L^2(M_q, \varphi) \ominus L^2(B_0, \varphi)_{B_0}$ is isomorphic to a sub-bimodule of $\oplus_{n=1}^{\mathbf{N}} L^2(M_{\xi_0}, \varphi) \otimes L^2(B_0, \varphi)$, where $\mathbf{N} \leq \aleph_0$.

Proof. Note that a linear combination of Hilbert-Schmidt operators is again a Hilbert-Schmidt operator. Thus, noting that $A^{\frac{1}{2}}\xi_{2p-1}$ and $A^{\frac{1}{2}}\xi_{2p}$ is a linear combination of ξ_{2p-1} and ξ_{2p} for all $1 \leq p \leq N_2$ (follows from Eq. (15) by using functional calculus), it follows that one may replace $\xi_{j_{l'}}$ by $A^{\frac{1}{2}}\xi_{j_{l'}}$ for $1 \leq l' \leq k$ in Lemma 3.1 to conclude the result.

The final statement follows from Eq. (6), Eq. (13), Prop. A.2, Prop. A.3 and the fact that span $\{\xi_{i_1} \otimes \cdots \otimes \xi_{i_m} : \xi_{i_l} \notin \{\xi_0, \xi_0'\}$ for at least one $l, 1 \leq l \leq m, m \in \mathbb{N}\}$ is dense in $L^2(M_q, \varphi) \ominus L^2(B_0, \varphi)$. □

We are now prepared to establish the factoriality of M_q in the case $dim(\mathcal{H}_{\mathbb{R}}) \geq 3$.

Theorem 3.3. *M_q is a factor if $dim(\mathcal{H}_{\mathbb{R}}) \geq 3$.*

Proof. If $dim(\mathcal{H}_{\mathbb{R}})$ is odd, or (U_t) has a nontrivial fixed vector, or (U_t) has a non zero weakly mixing component then the factoriality of M_q has been established in §6 of [BM16]. Thus, it remains to consider the case when $dim(\mathcal{H}_{\mathbb{R}}) \geq 4$, (U_t) is ergodic but not weakly mixing. Thus, assume that $dim(\mathcal{H}_{\mathbb{R}})$ is even (with ∞ regarded as an even number).

Let $x \in \mathcal{Z}(M_q)$ be a self-adjoint operator. Let $x_0 = x - \mathbb{E}_{B_0}(x)$. Note that both x and x_0 are analytic with respect to (σ_t^{φ}). Also note that $x_0 = x_0^*$. Then by Prop. A.3, Prop. A.4 and Thm. 3.2 it follows that if $u_n \in \mathcal{U}(M_{\xi_0})$ is sequence such that $u_n \to 0$, weakly, then $\left\|\mathbb{E}_{B_0}(x_0^* u_n x_0)\right\|_q \to 0$ as $n \to \infty$. But $u_n x_0 = u_n x - u_n \mathbb{E}_{B_0}(x) = u_n x - \mathbb{E}_{B_0}(u_n x) = x u_n - \mathbb{E}_{B_0}(x) u_n = x_0 u_n$ for all n. Therefore, $\left\|\mathbb{E}_{B_0}(x_0^* x_0)\right\|_q = \left\|u_n \mathbb{E}_{B_0}(x_0^* x_0)\right\|_q \to 0$ as $n \to 0$. As $\mathbb{E}_{B_0}$ is faithful, so $x_0 = 0$. Consequently, $x = \mathbb{E}_{B_0}(x)$. Therefore, $\mathcal{Z}(M_q) \subseteq B_0$.

Since $dim(\mathcal{H}_{\mathbb{R}}) \geq 4$ and is even, so there exists a pair $\zeta_0, \zeta_0' \in \mathcal{H}_{\mathbb{R}}$ of orthogonal analytic vectors (as before) arising from a direct summand in Eq. (14) which is different from the direct summand that corresponds to the pair ξ_0, ξ_0'. Let $\widetilde{B}_0 = vN(s_q(\zeta_0), s_q(\zeta_0'))$. Observe that B_0 and $\widetilde{B}_0$ are orthogonal (see [Po83]) with respect to the q-quasi-free state φ from Eq. (2). But the above argument shows that $\mathcal{Z}(M_q) \subseteq B_0 \cap \widetilde{B}_0$, whence the center is reduced to scalars, as was exactly required. □

The above analysis forces the relative commutant of M_{ξ_0} to be contained inside B_0.

Theorem 3.4. *$M_{\xi_0}' \cap M_q \subseteq B_0$.*

Proof. The proof is easy. Let $x \in M_{\xi_0}' \cap M_q$ be such that $\mathbb{E}_{B_0}(x) = 0$. Then $x\Omega$ is a (φ, φ)-bounded vector for ${}_{M_{\xi_0}}L^2(M_q, \varphi) \ominus L^2(B_0, \varphi)_{B_0}$. Consequently, (ii) of Thm. A.3 holds. Then by the proof of Thm. 3.3, we have $\left\|\mathbb{E}_{B_0}(x^* x)\right\|_q = \left\|\mathbb{E}_{B_0}(x^* u_n x)\right\|_q \to 0$ as $n \to \infty$, where $u_n \in \mathcal{U}(M_{\xi_0})$ is a sequence that goes to zero weakly as $n \to \infty$. It follows that $x = 0$ as $\mathbb{E}_{B_0}$ is faithful. The rest is similar to the first part of the argument of the proof of Thm. 3.3. □

Remark 3.5. Note that the approach to the proof of Thm. 3.3 subsumes the case that (U_t) has a fixed vector by appropriately modifying the subalgebra B_0 (c.f. [BM16]). So the approach to the proof of Thm. 3.3 is more general, as in this case we can overcome the non existence of conditional expectation onto M_{ξ_0}.

Appendix A. Bimodules

To decide the factoriality of $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$, $q \in (-1, 1)$, we need the machinery of bimodules. In this section, we record some facts on bimodules over von Neumann algebras that

are appropriate to our context. Most of what follows is well known and some are taken from [OOT15]. For comprehensive account on bimodules see [Co94, Po86, Ta03, Fa00]. We point out that many of the results in this section are known in greater generality but we state them in a way that is aligned to our set up. Readers familiar with the general theory of bimodules can skip this section. Thm. A.2 and Prop. A.4 (of this section) will be necessary for our purpose.

All Hilbert spaces in this section are separable and all von Neumann algebras have separable preduals. Following the tradition in the general theory of von Neumann algebras, in this section we assume that inner products are linear in the first variable. This change does not alter any conclusion and thus should cause no confusion.

Let M and N be von Neumann algebras. A Hilbert space $\mathcal{H}$ (also written as ${}_M\mathcal{H}_N$) is said to be a M-N Hilbert bimodule (or bimodule in short) if it is equipped with a $*$-representation $\pi_{\mathcal{H}}$ of $M \odot N^{\mathrm{op}}$ that is individually normal in each components, where N^{op} denotes the opposite von Neumann algebra of N; elements of N^{op} are written as y^{op} with $y \in N$. One refers $\pi_{\mathcal{H}|M}$ as the left M-action and $\pi_{\mathcal{H}|N^{\mathrm{op}}}$ as the right N-action, and write $x\xi y = \pi_{\mathcal{H}}(x \otimes y^{\mathrm{op}})\xi$, for $x \in M$, $y \in N$, and $\xi \in \mathcal{H}$. The inner product and norm on $\mathcal{H}$ will be denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ respectively.

Further, let φ and ψ be faithful normal states on M and N respectively. In this case, we suppose that M and N are in standard form acting on the GNS Hilbert spaces $\mathcal{H}_\varphi = L^2(M, \varphi)$ and $\mathcal{H}_\psi = L^2(N, \psi)$ respectively. Let $J_\varphi, \Omega_\varphi$ and J_ψ, Ω_ψ denote the canonical conjugation operator and the standard vacuum vector associated to $\mathcal{H}_\varphi$ and $\mathcal{H}_\psi$ respectively. The inner product and norm on the GNS space $\mathcal{H}_\varphi$ (resp. $\mathcal{H}_\psi$) will be denoted by $\langle \cdot, \cdot \rangle_\varphi$ and $\|\cdot\|_{2,\varphi}$ (resp. $\langle \cdot, \cdot \rangle_\psi$ and $\|\cdot\|_{2,\psi}$) respectively. Note that the identity bimodule over M is the M-M bimodule $\mathcal{H}_\varphi$, with left and right actions given by $x\xi y = xJ_\varphi y^* J_\varphi \xi$, $x, y \in M$ and $\xi \in \mathcal{H}$.

For M-N bimodules $\mathcal{H}$ and $\mathcal{K}$, the Banach space of bounded right N-module maps from $\mathcal{H}$ into $\mathcal{K}$ is denoted by $\mathbb{B}(\mathcal{H}_N, \mathcal{K}_N)$. In case $\mathcal{H} = \mathcal{K}$, we simply denote it by $\mathbb{B}(\mathcal{H}_N)$. Thus, $\mathbb{B}(\mathcal{H}_{\psi_N})$ coincides with N acting on $\mathcal{H}_\psi$ from the left.

Definition A.1. (i) A vector $\xi \in {}_M\mathcal{H}_N$ is said to be left ψ-bounded, if $L_\psi(\xi) : \mathcal{H}_\psi \to \mathcal{H}$ by $L_\psi(\xi)(J_\psi y^* J_\psi \Omega_\psi) = \xi y$, $y \in N$, is bounded and hence defines an element of $\mathbb{B}(\mathcal{H}_{\psi_N}, \mathcal{H}_N)$.
(ii) A vector $\xi \in {}_M\mathcal{H}_N$ is said to be right φ-bounded, if $R_\varphi(\xi) : \mathcal{H}_\varphi \to \mathcal{H}$ by $R_\varphi(\xi)(x\Omega_\varphi) = x\xi$, $x \in M$, is bounded.
(iii) A vector $\xi \in {}_M\mathcal{H}_N$ is said to be (φ, ψ)-bounded (or bounded), if it is both left ψ-bounded and right φ-bounded.

Note that the subspace $\mathfrak{D}(\mathcal{H}, \psi)$ of left ψ-bounded vectors is dense in $\mathcal{H}$ (see [Ta03, p. 188]). Further, if $\xi \in \mathfrak{D}(\mathcal{H}, \psi)$ and $x \in M$, then $x\xi \in \mathfrak{D}(\mathcal{H}, \psi)$ and $L_\psi(x\xi) = xL_\psi(\xi)$. Moreover, if $\xi_1, \xi_2 \in \mathfrak{D}(\mathcal{H}, \psi)$ then $L_\psi(\xi_2)^* L_\psi(\xi_1) \in \mathbb{B}(\mathcal{H}_{\psi_N}) = N$. Let $\mathfrak{D}(\mathcal{H}, \varphi, \psi)$ denote the collection of all (φ, ψ)-bounded vectors. Following [OOT15], write $L_\psi(\xi_2^* \times \xi_1) = L_\psi(\xi_2)^* L_\psi(\xi_1)$. Note that by Thm. 1 of [OOT15], $\mathfrak{D}(\mathcal{H}, \varphi, \psi)$ is dense in $\mathcal{H}$.

There is an intrinsic relation between cyclic bimodules over von Neumann algebras and normal completely positive maps. More precisely, let $\Phi : M \to N$ be a normal completely positive map. Given such a map, one performs the Stinespring construction to obtain a cyclic M-N bimodule $(\mathcal{H}_\Phi, \xi_\Phi)$ via separation and completion of $M \odot N$ (equivalently, $M \odot \mathcal{H}_\psi$) with respect to the sesquilinear form:

$$\langle x_1 \otimes_\Phi y_1, x_2 \otimes_\Phi y_2 \rangle_\Phi := \langle \Phi(x_2^* x_1)\Omega_\psi y_1, \Omega_\psi y_2 \rangle_\psi, \text{ equivalently,}$$
$$\langle x_1 \otimes_\Phi \zeta_1, x_2 \otimes_\Phi \zeta_2 \rangle_\Phi := \langle \Phi(x_2^* x_1)\zeta_1, \zeta_2 \rangle_\psi,$$

and the bimodule structure is defined by $x(x_0 \otimes_\Phi \zeta_0)y := xx_0 \otimes_\Phi \zeta_0 y$, for $x, x_0, x_1, x_2 \in M$ and $y, y_1, y_2 \in N$ and $\zeta_0, \zeta_1, \zeta_2 \in \mathcal{H}_\psi$. The distinguished unit cyclic vector ξ_Φ in $\mathcal{H}_\Phi$ is the class of $1_M \otimes 1_N$. There is also a converse to the above construction. We quote the following theorem from [OOT15] for the sake of convenience.

Theorem A.2. *Let M and N be von Neumann algebras equipped with faithful normal states φ and ψ respectively. Let ${}_M\mathcal{H}_N$ be a M-N bimodule and let $\xi \in \mathfrak{D}(\mathcal{H}, \varphi, \psi)$. Then, the map*

$$\Phi_\xi \colon M \ni x \mapsto L_\psi(\xi)^* x L_\psi(\xi) = L_\psi(\xi^* \times x\xi) \in N,$$

is normal, completely positive and satisfies $\psi \circ \Phi_\xi \leq \|R_\varphi(\xi)\|^2 \varphi$. The associated L^2-extension

$$T_{\Phi_\xi} \colon L^2(M,\varphi) \ni x\Omega_\varphi \mapsto \Phi_\xi(x)\Omega_\psi \in L^2(N,\psi),\ x \in M,$$

is equal to $L_\psi(\xi)^* R_\varphi(\xi)$. Further,

(22)
$$\langle \Phi_\xi(x)\Omega_\psi, J_\psi y^* J_\psi \Omega_\psi\rangle_\psi = \langle x\xi, \xi y\rangle_{\mathcal{H}} = \langle L_\psi(\xi)^* R_\varphi(\xi) x\Omega_\varphi, J_\psi y^* J_\psi \Omega_\psi\rangle_\psi,\ x \in M, y \in N.$$

In particular, the cyclic M-N bimodules $\mathcal{H}_{\Phi_\xi}$ and $\overline{M\xi N}$ are isomorphic via the association $x\xi y \mapsto x \otimes_{\Phi_\xi} J_\psi y^* \Omega_\psi$, for $x \in M$ and $y \in N$. Moreover, if T_{Φ_ξ} is a Hilbert-Schmidt operator then $\overline{M\xi N}$ as a M-N bimodule is isomorphic to a sub-bimodule of $L^2(M,\varphi) \otimes L^2(N,\psi)$.

Proof. We only prove the last statement for the sake of convenience. The rest is proved in [OOT15]. Let $T_{\Phi_\xi} = \sum_n \lambda_n \langle \cdot, \zeta_n\rangle_\varphi \delta_n$ denote the singular value decomposition of T_{Φ_ξ}, where $\zeta_n \in L^2(M,\varphi)$ and $\delta_n \in L^2(N,\psi)$ are unit vectors for all n (the sum above can have finitely many terms). By the hypothesis $\sum_n |\lambda_n|^2 < \infty$. Let $\eta = \sum_n \lambda_n (\zeta_n \otimes J_\psi \delta_n) \in L^2(M,\varphi) \otimes L^2(N,\psi)$.

Let $x_1, x_2 \in M$ and $y_1, y_2 \in N$. Note that the product in N^{op} is given by $y_1^{\mathrm{op}} y_2^{\mathrm{op}} = (y_2 y_1)^{\mathrm{op}}$. From Eq. (22) it follows that

$$\begin{aligned}
\langle x_1 \otimes_{\Phi_\xi} J_\psi y_1^* \Omega_\psi, x_2 \otimes_{\Phi_\xi} J_\psi y_2^* \Omega_\psi\rangle_{\Phi_\xi} &= \langle x_1 \xi y_1, x_2 \xi y_2\rangle_{\mathcal{H}} \\
&= \langle x_2^* x_1 \xi, \xi y_2 y_1^*\rangle_{\mathcal{H}} \\
&= \langle T_{\Phi_\xi}(x_2^* x_1 \Omega_\varphi), J_\psi y_1 y_2^* J_\psi \Omega_\psi\rangle_\psi \\
&= \sum_n \lambda_n \langle x_2^* x_1 \Omega_\varphi, \zeta_n\rangle_\varphi \langle \delta_n, J_\psi y_1 y_2^* J_\psi \Omega_\psi\rangle_\psi \\
&= \sum_n \lambda_n \langle x_2^* x_1 \Omega_\varphi, \zeta_n\rangle_\varphi \langle y_1 y_2^* \Omega_\psi, J_\psi \delta_n\rangle_\psi \\
&= \langle x_2^* x_1 \Omega_\varphi \otimes y_1 y_2^* \Omega_\psi, \eta\rangle_{\varphi\otimes\psi} \\
&= \langle x_2^* x_1 \Omega_\varphi \otimes (y_2^{\mathrm{op}})^* y_1^{\mathrm{op}} \Omega_\psi, \eta\rangle_{\varphi\otimes\psi}.
\end{aligned} \tag{23}$$

Thus, if $x_i \in M$ and $y_i \in N$ for $1 \leq i \leq n$, then from Eq. (23) one has

$$\begin{aligned}
0 &\leq \Big\langle \sum_i x_i \otimes_{\Phi_\xi} J_\psi y_i^* \Omega_\psi, \sum_j x_j \otimes_{\Phi_\xi} J_\psi y_j^* \Omega_\psi \Big\rangle_{\Phi_\xi} \\
&= \Big\langle \sum_{i,j} x_j^* x_i \Omega_\varphi \otimes (y_j^{\mathrm{op}})^* y_i^{\mathrm{op}} \Omega_\psi, \eta \Big\rangle_{\varphi\otimes\psi} \\
&= \Big\langle \Big(\sum_i x_i \otimes y_i^{\mathrm{op}}\Big)(\Omega_\varphi \otimes \Omega_\psi), \Big(\sum_j x_j \otimes y_j^{\mathrm{op}}\Big)\eta \Big\rangle_{\varphi\otimes\psi}.
\end{aligned} \tag{24}$$

Note that $\mathcal{M} = M \overline{\otimes} N^{\mathrm{op}}$ is in standard form on $L^2(M,\varphi) \otimes L^2(N,\psi)$ and $\mathfrak{A} = M\Omega_\varphi \otimes N^{\mathrm{op}}\Omega_\psi$ is a left Hilbert algebra associated with $\mathcal{M}$ (see Prop. 8.1 [St]), and $\mathfrak{A}'' = \mathcal{M}(\Omega_\varphi \otimes \Omega_\psi)$ is the full left Hilbert algebra associated to $\mathfrak{A}$. By Eq. (24) it follows that the right multiplication operator $R_\eta^0 : \mathfrak{A}'' \to L^2(M,\varphi) \otimes L^2(N,\psi)$ extending the map $(x\Omega_\varphi \otimes y^{\mathrm{op}}\Omega_\psi) \mapsto (x \otimes y^{\mathrm{op}})\eta$, for $x \in M$ and $y \in N$ is positive; this follows from the density of $M \odot N^{\mathrm{op}}$ in $\mathcal{M}$. Clearly, R_η^0 is affiliated to $\mathcal{M}'$. Consequently, the Friedrichs extension

$\widetilde{R}_\eta$ of R^0_η is positive, self-adjoint and affiliated to $\mathcal{M}'$ (see Ex. 9.27 [StZs], p. 239). Let $\varsigma = \widetilde{R}_\eta^{\frac{1}{2}}(\Omega_\varphi \otimes \Omega_\psi)$.

Finally, let $\overline{M\xi N} \ni x\xi y \mapsto (x \otimes y^{\mathrm{op}})\varsigma \in L^2(M,\varphi) \otimes L^2(N,\psi)$, for $x \in M$ and $y \in N$. Then for $u_1, u_2 \in \mathcal{U}(M)$ and $v_1, v_2 \in \mathcal{U}(N)$, using Eq. (23) one has

$$\begin{aligned}\langle u_1\xi v_1, u_2\xi v_2\rangle_{\mathcal{H}} &= \langle (u_1 \otimes v_1^{\mathrm{op}})(\Omega_\varphi \otimes \Omega_\psi), (u_2 \otimes v_2^{\mathrm{op}})\eta\rangle_{\varphi\otimes\psi} \\ &= \langle (u_1 \otimes v_1^{\mathrm{op}})(\Omega_\varphi \otimes \Omega_\psi), \widetilde{R}_\eta(u_2 \otimes v_2^{\mathrm{op}})(\Omega_\varphi \otimes \Omega_\psi)\rangle_{\varphi\otimes\psi} \\ &= \langle \widetilde{R}_\eta^{\frac{1}{2}}(u_1 \otimes v_1^{\mathrm{op}})(\Omega_\varphi \otimes \Omega_\psi), \widetilde{R}_\eta^{\frac{1}{2}}(u_2 \otimes v_2^{\mathrm{op}})(\Omega_\varphi \otimes \Omega_\psi)\rangle_{\varphi\otimes\psi} \\ &= \langle (u_1 \otimes v_1^{\mathrm{op}})\widetilde{R}_\eta^{\frac{1}{2}}(\Omega_\varphi \otimes \Omega_\psi), (u_2 \otimes v_2^{\mathrm{op}})\widetilde{R}_\eta^{\frac{1}{2}}(\Omega_\varphi \otimes \Omega_\psi)\rangle_{\varphi\otimes\psi} \\ &= \langle (u_1 \otimes v_1^{\mathrm{op}})\varsigma, (u_2 \otimes v_2^{\mathrm{op}})\varsigma\rangle_{\varphi\otimes\psi}.\end{aligned}$$

It follows that $\overline{M\xi N}$ and $\overline{(M\overline{\otimes}N^{\mathrm{op}})\varsigma}$ are isomorphic as M-N bimodules, the latter by construction is contained in $L^2(M,\varphi) \otimes L^2(N,\psi)$. $\square$

As a corollary of Thm. A.2 we obtain the following.

Proposition A.3. *Let M be a von Neumann algebra acting in standard form with respect to a faithful normal state φ. Let A, B be two unital von Neumann subalgebras of M and suppose that there is a φ-preserving faithful normal conditional expectation $\mathbb{E}_B$ from M onto B. Let $x \in M$ be analytic with respect to the modular automorphism group (σ_t^φ). Regarding $L^2(M,\varphi)$ as a A-B bimodule one has:*
(i) $x\Omega_\varphi$ is (φ,φ)-bounded.
(ii) $\Phi_{x\Omega_\varphi}(a) = \mathbb{E}_B(x^*ax)$, $a \in A$.

Proof. (i). Note that by the fundamental theorem of Tomita-Takesaki theory, one has

$$L_\varphi(x\Omega_\varphi)(J_\varphi y^* J_\varphi \Omega_\varphi) = J_\varphi y^* J_\varphi x\Omega_\varphi = xJ_\varphi y^* J_\varphi \Omega_\varphi, \text{ for all } y \in B.$$

Thus, $x\Omega_\varphi$ is left φ-bounded. Again, for $a \in A$ one has

$$R_\varphi(x\Omega_\varphi)(a\Omega_\varphi) = ax\Omega_\varphi = J_\varphi(\sigma^\varphi_{\frac{i}{2}}(x))^* J_\varphi a\Omega_\varphi, \; a \in A \text{ (see [Fa00])}.$$

Thus, $x\Omega_\varphi$ is right φ-bounded as well.
(ii). Since $\mathbb{E}_B : M \to B$ exists, it follows that B is invariant under the modular automorphism group (σ_t^φ) [Ta72]. Thus if M_∞ denotes the collection of elements in M that are analytic with respect to (σ_t^φ), then $M_\infty \cap B$ is ultraweakly (also strongly) dense in B. Therefore, if $y_1, y_2 \in M_\infty \cap B$, then by Thm. A.2, Def. A.1 and [Fa00], we have

$$\begin{aligned}&\langle L_\varphi(x\Omega_\varphi)^* a L_\varphi(x\Omega_\varphi)(J_\varphi y_1^* J_\varphi \Omega_\varphi), (J_\varphi y_2^* J_\varphi \Omega_\varphi)\rangle_\varphi \\ =&\langle L_\varphi(x\Omega_\varphi)^* L_\varphi(ax)(J_\varphi y_1^* J_\varphi \Omega_\varphi), (J_\varphi y_2^* J_\varphi \Omega_\varphi)\rangle_\varphi \\ =&\langle L_\varphi(ax\Omega_\varphi)(J_\varphi y_1^* J_\varphi \Omega_\varphi), L_\varphi(x\Omega_\varphi)(J_\varphi y_2^* J_\varphi \Omega_\varphi)\rangle_\varphi \\ =&\langle J_\varphi y_1^* J_\varphi ax\Omega_\varphi, J_\varphi y_2^* J_\varphi x\Omega_\varphi\rangle_\varphi \\ =&\langle axJ_\varphi y_1^* J_\varphi \Omega_\varphi, xJ_\varphi y_2^* J_\varphi \Omega_\varphi\rangle_\varphi \\ =&\langle (x^*ax)J_\varphi y_1^* J_\varphi \Omega_\varphi, J_\varphi y_2^* J_\varphi \Omega_\varphi\rangle_\varphi \\ =&\langle (x^*ax)\sigma^\varphi_{-\frac{i}{2}}(y_1)\Omega_\varphi, \sigma^\varphi_{-\frac{i}{2}}(y_2)\Omega_\varphi\rangle_\varphi \\ =&\varphi\left((\sigma^\varphi_{-\frac{i}{2}}(y_2))^*(x^*ax)\sigma^\varphi_{-\frac{i}{2}}(y_1)\right) \\ =&\varphi\left(\mathbb{E}_B\left((\sigma^\varphi_{-\frac{i}{2}}(y_2))^*(x^*ax)\sigma^\varphi_{-\frac{i}{2}}(y_1)\right)\right) \\ =&\varphi\left((\sigma^\varphi_{-\frac{i}{2}}(y_2))^*\mathbb{E}_B(x^*ax)\sigma^\varphi_{-\frac{i}{2}}(y_1)\right)\end{aligned}$$

$$=\langle \mathbb{E}_B(x^*ax)(J_\varphi y_1^* J_\varphi \Omega_\varphi), (J_\varphi y_2^* J_\varphi \Omega_\varphi)\rangle_\varphi.$$

By density it follows that

$$\Phi_{x\Omega_\varphi}(a) = L_\varphi(x\Omega_\varphi)^* a L_\varphi(x\Omega_\varphi) = \mathbb{E}_B(x^*ax), \ a \in A.$$

This completes the proof. □

Thus we have the following.

Proposition A.4. *Let M be a von Neumann algebra acting in standard form with respect to a faithful normal state φ. Let A, B be two unital von Neumann subalgebras of M such that $A \subseteq B$ and such that there exists a faithful normal φ-preserving conditional expectation $\mathbb{E}_B : M \to B$. Suppose that*

$$_AL^2(M,\varphi) \ominus L^2(B,\varphi)_B \subseteq \oplus_{n=1}^{\mathbf{N}} {}_AL^2(A,\varphi) \otimes L^2(B,\varphi)_B, \ \textit{for some } \mathbf{N} \leq \aleph_0.$$

Let $x \in M$ be analytic with respect to the modular automorphism group (σ_t^φ) and such that $\mathbb{E}_B(x) = 0$ and $\|x\| \leq 1$. Then

$$\widehat{\Phi}_{x\Omega_\varphi} : A \to L^2(B,\varphi) \ \textit{by} \ \Phi_{x\Omega_\varphi}(a) = \mathbb{E}_B(x^*ax)\Omega_\varphi, \ a \in A,$$

is a compact operator.

Proof. First of all note that B is globally invariant under (σ_t^φ) by a well known theorem of Takesaki [Ta72]. So, the modular group and the modular conjugation of B with respect to the restricted state $\varphi_{|B}$ is the restriction of (σ_t^φ) and J_φ respectively to B and $L^2(B,\varphi)$. Thus, $L^2(M,\varphi) \ominus L^2(B,\varphi)$ is a A-B bimodule with natural actions. Indeed, the analytic elements of B are *s.o.t.* dense in B. Thus, by [Fa00] it follows that if $b \in B$ is analytic, then $J_\varphi b^* J_\varphi y\Omega_\varphi = y\sigma^\varphi_{-\frac{i}{2}}(b)\Omega_\varphi$ for $y \in M$. This forces that $L^2(M,\varphi) \ominus L^2(B,\varphi)$ is a right B-module. That $L^2(M,\varphi) \ominus L^2(B,\varphi)$ is a left A-module is obvious. The rest of the objects in the statement are thus well defined from Prop. A.3.

The conclusion is a standard fact on coarse bimodules and is well known to experts. □

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